CHAPTER 7

Gamma, Beta Function and their Properties, Dirichlet's Integral and its Applications

7.1 INTRODUCTION

Gamma function $\overline{|(z)}$ is widely used in the mathematical and applied sciences, similar to well-known factorial symbol $n!$. L. Euler (1729) the famous mathematician, introduced it as a natural extension of the factorial operation $n!$ from positive integers n to real and even complex values of the argument n. This relation is described by the following formula:

7.2 GAMMA FUNCTION

(U.P. I Semester Dec. 2007)

Properties of Gamma function is very important function in mathematics and statistics. Gamma function is a continuous extension to the factorial function, which is only defined for the nonnegative integers. While there are other continuous extensions to the factorial function, the gamma function is the only one that is convex for positive real numbers.

The gamma function satisfies the recursive property which can be proved using integration by parts and L'Hopital's Rule.

$$\int_0^\infty e^{-x}x^{n-1}dx \qquad (n > 0)$$

is called gamma function on n. It is also written as $\overline{|n} = \int_0^\infty e^{-x}x^{n-1}dx$

$$\boxed{\int_0^\infty e^{-x}x^{n-1}dx = \overline{|n}}$$

Example 1. *Prove that* $\overline{|1} = 1$

Solution. We know that, $\overline{|n} = \int_0^\infty e^{-x}x^{n-1}dx$

Put $n = 1$, $\overline{|1} = \int_0^\infty e^{-x}x^{1-1}dx = \int_0^\infty e^{-x}dx = \left[\frac{e^{-x}}{-1}\right]_0^\infty = 1$ **Proved.**

Example 2. *Prove that*

(i) $\overline{|n+1} = n\overline{|n}$ (ii) $\overline{|n+1} = n!$ ***(Reduction formula)***

Solution.

(i) We know that, $\overline{|n} = \int_0^\infty x^{n-1}e^{-x}dx$ ... (1)

Integrating by parts, we have

$$\Gamma(n) = \left[x^{n-1}\frac{e^{-x}}{-1}\right]_0^\infty - (n-1)\int_0^\infty x^{n-2}\frac{e^{-x}}{-1}dx$$

$$= \lim_{x\to 0}\left\{\left(1-\frac{x}{1!}+\frac{x^2}{2!}-\ldots+(-1)^n\frac{x^n}{n!}+\ldots+\infty\right)x^{n-1}\right\}+(n-1)\int_0^\infty x^{n-2}e^{-x}dx = 0+(n-1)\int_0^\infty x^{n-2}e^{-x}dx$$

$$\Gamma(n) = (n-1)\Gamma(n-1) \qquad \ldots (2)$$

$\therefore$ $\boxed{\Gamma(n+1) = n\Gamma(n)}$ Replacing n by $(n+1)$ **Proved.**

(*ii*) Replacing n by $n-1$ in (2), we get

$$\Gamma(n-1) = (n-2)\Gamma(n-2)$$

Putting the value of $\Gamma(n-1)$ in (2), we get

$$\Gamma(n) = (n-1)(n-2)\Gamma(n-2)$$

Similarly,

$$\Gamma(n) = (n-1)(n-2).....3.2.1\Gamma(1) \qquad \ldots (3)$$

Putting the value of $\Gamma(1)$ in (3), we have

$$\Gamma(n) = (n-1)(n-2).....3.2.1.1$$

$$\Gamma(n) = (n-1)!$$

Replacing n by $n+1$ we have **Proved.**

$$\Gamma(n+1) = n!$$

Example 3. *Evaluate* $\Gamma\left(\frac{-5}{2}\right)$ (*MTU 2014*)

Solution. $\Gamma\left(-\frac{1}{2}+1\right) = -\frac{1}{2}\Gamma\left(-\frac{1}{2}\right)$

$\Rightarrow$ $\Gamma\left(\frac{1}{2}\right) = -\frac{1}{2}\Gamma\left(-\frac{3}{2}+1\right)$ $\Rightarrow$ $\sqrt{\pi} = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\Gamma\left(-\frac{3}{2}\right)$

$\Rightarrow$ $\sqrt{\pi} = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\Gamma\left(-\frac{5}{2}+1\right)$ $\Rightarrow$ $\sqrt{\pi} = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\Gamma\left(-\frac{5}{2}\right)$

$$\sqrt{\pi} = -\frac{15}{8}\Gamma\left(-\frac{5}{2}\right)$$

$$\Gamma\left(-\frac{5}{2}\right) = -\frac{8}{15}\sqrt{\pi}$$ **Ans.**

Example 4. *Evaluate* $\int_0^\infty \sqrt[4]{x}e^{-\sqrt{x}}dx$

Solution. Let $I = \int_0^\infty x^{1/4}e^{-\sqrt{x}}dx$ ... (1)

Putting $\sqrt{x} = t \Rightarrow x = t^2$ so that $dx = 2t\,dt$ in (1) we get

$$I = \int_0^\infty t^{1/2}e^{-t}2t\,dt = 2\int_0^\infty t^{3/2}e^{-t}dt = 2\int_0^\infty t^{\frac{5}{2}-1}e^{-t}dt$$

$$= 2\Gamma\left(\frac{5}{2}\right)$$ [By definition]

$$= 2\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right) = 2\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{2}\sqrt{\pi}$$ **Ans.**

Example 5. *Evaluate* $\int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}}\, dx.$

Solution. Let
$$I = \int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}}\, dx \qquad \ldots (1)$$

Putting $\sqrt[3]{x} = t \Rightarrow x = t^3$ so that $dx = 3t^2\, dt$ in (1), we get

$$I = \int_0^\infty t^{3/2} e^{-t}\, 3t^2\, dt = 3\int_0^\infty t^{7/2} e^{-t}\, dt = 3\int_0^\infty t^{\frac{9}{2}-1} e^{-t}\, dt = 3\Gamma\left(\frac{9}{2}\right) = 3\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{315}{16}\sqrt{\pi}$$ **Ans.**

Example 6. *Evaluate* $\int_0^\infty x^{n-1} e^{-h^2x^2}\, dx.$

Solution. Let
$$I = \int_0^\infty x^{n-1} e^{-h^2x^2}\, dx \qquad \ldots (1)$$

Putting $t = h^2 x^2 \Rightarrow x = \dfrac{\sqrt{t}}{h}$ so that $dx = \dfrac{dt}{2h\sqrt{t}}$, we get

$$I = \int_0^\infty \left(\frac{\sqrt{t}}{h}\right)^{n-1} e^{-t}\, \frac{dt}{2h\sqrt{t}} = \frac{1}{2h^n}\int_0^\infty t^{\frac{n-1}{2}} e^{-t}\, \frac{dt}{\sqrt{t}} = \frac{1}{2h^n}\int_0^\infty t^{\frac{n-2}{2}} e^{-t}\, dt$$

$$= \frac{1}{2h^n}\Gamma\left(\frac{n}{2}\right)$$ **Ans.**

Example 7. *Evaluate* $\int_0^\infty \dfrac{x^a}{a^x}\, dx.$ $(a > 1)$

Solution. Let
$$I = \int_0^\infty \frac{x^a}{a^x}\, dx \qquad \ldots (1)$$

Putting $a^x = e^t \Rightarrow x\log a = t \Rightarrow x = \dfrac{1}{\log a} \Rightarrow dx = \dfrac{dt}{\log a}$ in (1), we have

$$I = \int_0^\infty \left(\frac{t}{\log a}\right)^a e^{-t}\, \frac{dt}{\log a} = \frac{1}{(\log a)^{a+1}}\int_0^\infty e^{-t} t^a\, dt$$

$$= \frac{1}{(\log a)^{a+1}}\int_0^\infty t^{(a+1)-1} e^{-t}\, dt$$

$$= \frac{1}{(\log a)^{a+1}}\Gamma(a+1)$$ **Ans.**

Example 8. *To prove that*

$$\int_0^1 x^m (\log x)^n\, dx = \frac{(-1)^n}{(m+1)^{n+1}}\Gamma(n+1)$$

Proof. Put $\log x = -t$ so that $x = e^{-t} \Rightarrow dx = -e^{-t}\, dt$

$$x^m = e^{-mt}$$

$$(\log x)^n = (-t)^n$$

Now, $$\int_0^1 x^m (\log x)^n\,dx = \int_\infty^0 e^{-mt}(-t)^n(-e^{-t})\,dt = \int_0^\infty (-1)^n e^{-mt-t} t^n\,dt$$

Putting $(m+1)\,t = u$ so that $(m+1)\,dt = du$, we get

$$I = \int_0^\infty (-1)^n e^{-u}\cdot\frac{u^n}{(m+1)^n}\frac{du}{(m+1)}$$

$$= \frac{(-1)^n}{(m+1)^{n+1}}\int_0^\infty e^{-u}u^n\,du = \frac{(-1)^n}{(m+1)^{n+1}}\int_0^\infty e^{-u}u^{(n+1)-1}\,du$$

$$= \frac{(-1)^n}{(m+1)^{n+1}}\,\Gamma(n+1)$$ **Proved.**

Example 9. *Prove that* $\int_0^1 (x\log x)^4\,dx = \frac{4!}{5^5}$ (M.U. II Semester, 2009)

Solution. We know that

$$\int_0^1 x^m (\log x)^n\,dx = \frac{(-1)^n}{(m+1)^{n+1}}\cdot\Gamma(n+1)$$

Now, $$\int_0^1 (x\log x)^4\,dx = \int_0^1 x^4(\log x)^4\,dx$$

Putting $m = n = 4$ in (1), we get

$$\int_0^1 x^4(\log x)^4\,dx = \frac{(-1)^4}{(4+1)^{4+1}}\Gamma(4+1)$$

$$= \frac{\Gamma(5)}{5^5} = \frac{4!}{5^5}$$ **Proved.**

Example 10. *Evaluate* $\int_0^1 x^{n-1}\cdot\left[\log_e\left(\frac{1}{x}\right)\right]^{m-1}\cdot dx$

Solution. Put $\log_e\frac{1}{x} = t$ or $x = e^{-t}$ $\therefore dx = -e^{-t}dt$

$$\int_0^1 x^{n-1}\cdot\left[\log_e\left(\frac{1}{x}\right)\right]^{m-1} dx = \int_\infty^0 (e^{-t})^{n-1}[t]^{m-1}(-e^{-t}t) = \int_0^\infty e^{-nt}t^{m-1}dt$$

Putting $nt = u \Rightarrow t = \frac{u}{n}$ so that $dt = \frac{du}{n}$, we get

$$\int_0^1 x^{n-1}\cdot\left[\log_e\left(\frac{1}{x}\right)\right]^{m-1} dx = \int_0^\infty e^{-u}\left(\frac{u}{n}\right)^{m-1}\frac{du}{n} = \frac{1}{n^m}\int_0^\infty e^{-u}u^{m-1}du = \frac{1}{n^m}\Gamma(m)$$ **Ans.**

7.3 TRANSFORMATION OF GAMMA FUNCTION

Prove that (*i*) $\int_0^\infty e^{-ky}y^{n-1}dy = \frac{\Gamma(n)}{k^n}$ (*ii*) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

(*iii*) $\int_0^1\left(\log\frac{1}{y}\right)^{n-1}dy = \Gamma(n)$ (*iv*) $\Gamma(n) = \frac{1}{n}\int_0^\infty e^{-x^{\frac{1}{n}}}dx$

Solution. We know that $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx$ ... (1)

(*i*) Replace x by $k\,y$, so that $dx = k\,dy$; then (1) becomes

$$\Gamma n = \int_0^\infty (k\,y)^{n-1} e^{-ky} k\,dy.$$

$$\Gamma n = k^n \int_0^\infty e^{-ky} y^{n-1} dy$$

$$\boxed{\int_0^\infty e^{-ky} y^{n-1} dy = \frac{\Gamma n}{k^n}} \qquad \text{... (2)}$$ **Proved.**

(*ii*) Replace x^n by y, so that $nx^{n-1}\,dx = dy$ in (1), then

$$\Gamma n = \int_0^\infty y^{\frac{n-1}{n}} e^{-y^{1/n}} \frac{dy}{nx^{n-1}}$$

$$= \int_0^\infty y^{\frac{n-1}{n}} e^{-y^{1/n}} \frac{dy}{ny^{\frac{n-1}{n}}} = \frac{1}{n}\int_0^\infty e^{-y^{1/n}} dy$$

When $n = \frac{1}{2}$,

$$\Gamma\frac{1}{2} = \frac{1}{\frac{1}{2}}\int_0^\infty e^{-y^2} dy = 2\left[\frac{1}{2}\sqrt{\pi}\right]$$

$$\boxed{\Gamma\frac{1}{2} = \sqrt{\pi}}$$ **Proved.**

(*iii*) Putting $e^{-x} = y$, so that $-e^{-x}\,dx = dy$ and $-x = \log y$, $x = \log\frac{1}{y}$, (1) becomes

$$\Gamma n = -\int_1^0 \left(\log\frac{1}{y}\right)^{n-1} y\cdot\frac{dy}{e^{-x}}$$

$$= \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} y\cdot\frac{dy}{y} = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy$$ **Proved.**

(*iv*) We know that, $\Gamma(n) = \int_0^\infty e^{-x}\cdot x^{n-1} dx$ ... (1)

Putting $x^n = y \Rightarrow x = y^{1/n}$ so that $dx = \frac{1}{n} y^{\frac{1}{n}-1} dy$ in (1), we get

$$\Gamma n = \int_0^\infty e^{-y^{1/n}} y^{\frac{n-1}{n}} \cdot \frac{1}{n} y^{\frac{1}{n}-1} dy = \frac{1}{n}\int_0^\infty e^{-y^{\frac{1}{n}}} dy$$

$$\Gamma n = \frac{1}{n}\int_0^\infty e^{-x^{\frac{1}{n}}} dx.$$ **Proved.**

EXERCISE 7.1

Evaluate:

1. (*i*) $\Gamma\left(-\frac{3}{2}\right)$ (*ii*) $\Gamma\left(-\frac{15}{2}\right)$ (*iii*) $\Gamma\frac{7}{2}$ (*iv*) $\Gamma 0$

Ans. (*i*) $\frac{4}{3}\sqrt{\pi}$ (*ii*) $\frac{2^8\sqrt{\pi}}{15\times13\times11\times9\times7\times5\times3}$ (*iii*) $\frac{15\sqrt{\pi}}{8}$ (*iv*) ∞

2. $\int_0^\infty \sqrt{x}e^{-x}dx$ (*GBTU 2014*) **Ans.** $\Gamma\frac{3}{2}$

Ans. $\frac{3\sqrt{\pi}}{8}$

3. $\int_0^\infty x^4 e^{-x^2} dx$ (UP. Jan. 2011) Ans. $\frac{\sqrt{\pi}}{2h}$

4. $\int_0^\infty e^{-h^2x^2} dx$ Ans. $\frac{\Gamma m \Gamma n}{4a^m b^n}$

5. $\int_0^\infty \int_0^\infty e^{-(ax^2+by^2)} x^{2m-1} y^{2n-1} dx dy,\ a,b,m,n>0$ Ans. $\sqrt{\pi}$

6. $\int_0^1 \frac{dx}{\sqrt{-\log x}}$ Ans. $-\frac{3}{128}$

7. $\int_0^1 (x\log x)^3 dx$ Ans. $\sqrt{2\pi}$

8. $\int_0^1 \frac{dx}{\sqrt{x\log\frac{1}{x}}}$

9. Prove that $1.3.5....(2n-1) = \frac{2^n \Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}}$

10. $\int_0^\infty e^{-y^{1/m}} dy = m\Gamma m$

11. $\int_0^\infty \int_0^\infty e^{-(ax^2+by^2)} x^{2m-1} y^{2n-1} dx dy = \frac{\Gamma(m)\Gamma(n)}{4a^m b^n}$, where a, b, m, n are positive.

12. $\int_0^{\pi/2} \frac{d\theta}{(a\cos^4\theta + b\sin^4\theta)} = \frac{\left(\Gamma(1/4)\right)^2}{4(ab)^{1/4}\sqrt{\pi}}$ [**Hint.** Put $\tan\theta = t$ then $bt^4 = az$]

(U.P. I Semester Dec. 2007)

7.4 BETA FUNCTION

$$\int_0^1 x^{l-1}(1-x)^{m-1} dx \qquad (l>0,\ m>0)$$

is called the Beta function of l, m. It is also written as

$$\boxed{\beta(l,m) = \int_0^1 x^{l-1}(1-x)^{m-1} dx}$$

7.5 EVALUATION OF BETA FUNCTION

$$\beta(l,m) = \frac{\Gamma l\,\Gamma m}{\Gamma(l+m)}$$

Solution. We have, $\beta(l,m) = \int_0^1 x^{l-1}(1-x)^{m-1} dx = \int_0^1 (1-x)^{m-1} x^{l-1} dx$

Integration by parts, we have

$$= \left[(1-x)^{m-1}\frac{x^l}{l}\right]_0^1 + (m-1)\int_0^1 (1-x)^{m-2}\left(\frac{x^l}{l}\right) dx$$

$$= \frac{(m-1)}{l}\int_0^1 (1-x)^{m-2} x^l dx$$

Again integrating by parts, we get

$$= \frac{(m-1)(m-2)}{l(l+1)}\int_0^1 (1-x)^{m-3} x^{l+1} dx$$

$$= \frac{(m-1)(m-2)...2.1}{l(l+1).....(l+m-2)} \int_0^1 x^{l+m-2} dx$$

$$= \frac{(m-1)(m-2)...2.1}{l(l+1).....(l+m-2)} \left[\frac{x^{l+m-1}}{l+m-1} \right]_0^1$$

$$= \frac{(m-1)(m-2)...2.1}{l(l+1).....(l+m-2)(l+m-1)}$$

$$= \frac{(m-1)!}{l(l+1).....(l+m-2)(l+m-1)} \times \frac{(l-1)(l-2).....1}{(l-1)(l-2).....1}$$

$$= \frac{(m-1)!\,(l-1)!}{1.2....(l-2)(l-1)\cdot l(l+1)...(l+m-2)(l+m-1)}$$

$$= \frac{(l-1)!(m-1)!}{(l+m-1)!}$$

$$\boxed{\beta(l,m) = \frac{\Gamma l\,\Gamma m}{\Gamma(l+m)}}$$

And if only l is positive integer and not m then

$$\boxed{\beta(l,m) = \frac{(l-1)!}{m(m+1)...(m+l-1)}}$$ **Ans.**

7.6 A PROPERTY OF BETA FUNCTION

$$\beta(l,\ m) = \beta(m,\ l)$$

Solution. We have

$$\beta(l,m) = \int_0^1 x^{l-1}(1-x)^{m-1}\,dx \qquad \left[\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^1 (1-x)^{l-1}[1-(1-x)]^{m-1} dx$$

$$= \int_0^1 (1-x)^{l-1}\, x^{m-1}\, dx$$

$$= \int_0^1 x^{m-1}(1-x)^{l-1}\, dx$$

l and m are interchanged

$$\boxed{\beta(l,\ m) = \beta(m,\ l)}$$ **Proved.**

Example 11. *Evaluate* $\int_0^1 x^4\left(1-\sqrt{x}\right)^5 dx$

Solution. Let $\sqrt{x} = t \Rightarrow x = t^2$ so that $dx = 2t\,dt$

$$\int_0^1 x^4\left(1-\sqrt{x}\right)^5 dx = \int_0^1 (t^2)^4(1-t)^5(2t\,dt)$$

$$= 2\int_0^1 t^9(1-t)^5\,dt = 2\beta(10,6) = 2\frac{\Gamma 10\,\Gamma 6}{\Gamma 16} = 2\cdot\frac{9!5!}{(15)!}$$

$$= 2\cdot\frac{5!}{10\times11\times12\times13\times14\times15} = \frac{2\times1\times2\times3\times4\times5}{10\times11\times12\times13\times14\times15}$$

$$= \frac{1}{11\times13\times7\times15} = \frac{1}{15015}$$ **Ans.**

Example 12. *Evaluate* $\int_0^1 (1-x^3)^{-\frac{1}{2}}\,dx$

Solution. Let $x^3 = y \Rightarrow x = y^{1/3}$ so that $dx = \frac{1}{3}y^{-\frac{2}{3}}\,dy$

$$\int_0^1 (1-x^3)^{-\frac{1}{2}}\,dx = \int_0^1 (1-y)^{-\frac{1}{2}}\left(\frac{1}{3}y^{-\frac{2}{3}}\,dy\right)$$

$$= \frac{1}{3}\int_0^1 y^{-\frac{2}{3}}(1-y)^{-\frac{1}{2}}\,dy = \frac{1}{3}\beta\left(\frac{1}{3},\frac{1}{2}\right) = \frac{1}{3}\,\frac{\Gamma\frac{1}{3}\,\Gamma\frac{1}{2}}{\Gamma\frac{5}{6}}$$ **Ans.**

7.7 TRANSFORMATION OF BETA FUNCTION

We know that

$$\beta(l,m) = \int_0^1 x^{l-1}(1-x)^{m-1}\,dx, \qquad \ldots (1)$$

Putting $x = \frac{1}{1+y}$ so that $dx = -\frac{1}{(1+y)^2}dy$ and $1-x = \frac{y}{1+y}$ in (1), we get

$$\beta(l,m) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{l-1}\left(\frac{y}{1+y}\right)^{m-1}\left[-\frac{1}{(1+y)^2}dy\right]$$

$$= \int_0^\infty \frac{y^{m-1}}{(1+y)^{l+m}}dy$$

Since l, m can be interchanged in $\beta(l, m)$,

$$\beta(l,m) = \int_0^\infty \frac{y^{l-1}}{(1+y)^{m+l}}dy \Rightarrow \boxed{\beta(l,m) = \int_0^\infty \frac{x^{l-1}}{(1+x)^{m+l}}dx}$$

Example 13. *Evaluate* $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$ [*MTU, 2012*]

Solution. We know that

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$$

$$\Rightarrow \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx + \int_1^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx = \beta(m,n) \qquad \ldots (1)$$

Consider $\int_1^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx = \int_1^0 \frac{\left(\frac{1}{t}\right)^{n-1}}{\left(1+\frac{1}{t}\right)^{m+n}}\left(-\frac{1}{t^2}dt\right) = \int_0^1 \frac{\left(\frac{1}{t}\right)^{n-1}\frac{1}{t^2}}{\left(\frac{1}{t}\right)^{m+n}(t+1)^{m+n}}dt$, $\left(\text{Put } x = \frac{1}{t}\right)$

$$= \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}} dt = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

Putting the value of $\int_1^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$ in (1), we get

$$\int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx = \beta(m,n)$$

$\Rightarrow$ $$\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} dx = \beta(m,n)$$ **Ans.**

7.8 RELATION BETWEEN BETA AND GAMMA FUNCTIONS

We know that, $\quad \overline{|l} = \int_0^\infty e^{-x} x^{l-1}\, dx,$ [Put $zx = y$]

$$\frac{\overline{|l}}{z^l} = \int_0^\infty e^{-zx} x^{l-1} dx$$

$$\overline{|l} = \int_0^\infty z^l e^{-zx} x^{l-1}\, dx$$

Multiplying both sides by $e^{-z} z^{m-1}$, we have

$$\overline{|l} \cdot e^{-z} \cdot z^{m-1} = \int_0^\infty e^{-z} \cdot z^{m-1} \cdot z^l \cdot e^{-zx} \cdot x^{l-1} dx$$

$$\overline{|l} \cdot e^{-z} \cdot z^{m-1} = \int_0^\infty e^{-(1+x)z} \cdot z^{l+m-1} \cdot x^{l-1} dx$$

Integrating both sides w.r.t. 'z', we get

$$\int_0^\infty \sqrt{l} e^{-z} z^{m-1}\, dz = \int_0^\infty \int_0^\infty e^{-(1+x)z} z^{l+m-1} x^{l-1}\, dx\, dz$$

$$\overline{|l}\,\overline{|m} = \int_0^\infty x^{l-1}\, dx \int_0^\infty e^{-(1+x)z} z^{l+m-1}\, dz$$

$$= \int_0^\infty x^{l-1}\, dx \cdot \frac{\overline{|l+m}}{(1+x)^{l+m}}$$

$$\overline{|l}\,\overline{|m} = \overline{|l+m} \int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}} dx = \overline{|l+m} \cdot \beta(l,m)$$

$$\therefore \quad \boxed{\beta(l,m) = \frac{\overline{|l}\,\overline{|m}}{\overline{|l+m}}}$$

This is the required relation.

7.9 SHOW THAT

$$\boxed{\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta\, d\theta = \frac{\overline{\left|\left(\frac{p+1}{2}\right)\right.}\,\overline{\left|\left(\frac{q+1}{2}\right)\right.}}{2\overline{\left|\left(\frac{p+q+2}{2}\right)\right.}}}$$

Solution. We know that

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx \qquad \dots (1)$$

Putting $\quad x = \sin^2\theta,\ dx = 2\sin\theta\cos\theta\, d\theta$

and $\quad 1 - x = 1 - \sin^2\theta = \cos^2\theta$

Then (1) becomes

$$\beta(m,n)=\int_0^{\frac{\pi}{2}}\sin^{2m-2}\theta\cos^{2n-2}\theta\,2\sin\theta\cos\theta\,d\theta$$

or

$$\frac{\Gamma m\,\Gamma n}{\Gamma m+n}=2\int_0^{\frac{\pi}{2}}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$$

Putting $2m-1=p,\quad i.e.,\quad m=\dfrac{p+1}{2}$

and $2n-1=q,\quad i.e.,\quad n=\dfrac{q+1}{2}$

$$\frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{\Gamma\frac{p+q+2}{2}}=2\int_0^{\frac{\pi}{2}}\sin^p\theta\cdot\cos^q\theta\,d\theta$$

$$\boxed{\int_0^{\frac{\pi}{2}}\sin^p\theta\cos^q\theta\,d\theta=\frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}}$$

Proved.

Example 14. *Find the value of* $\Gamma\frac{1}{2}$.

Solution. We have already solved this problem in Art. 7.3 (ii) Transformation of the Gamma Function.

Now, by **Second method:** We know that,

$$\int_0^{\frac{\pi}{2}}\sin^p\theta\cdot\cos^q\theta\,d\theta=\frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$$

Putting $p=q=0$, we get $\displaystyle\int_0^{\frac{\pi}{2}}d\theta=\frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{2\,\Gamma 1}$

$$\Rightarrow\qquad[\theta]_0^{\pi/2}=\frac{1}{2}\left(\Gamma\frac{1}{2}\right)^2\;\Rightarrow\;\frac{\pi}{2}=\frac{1}{2}\left(\Gamma\frac{1}{2}\right)^2$$

$$\Rightarrow\qquad\left(\Gamma\frac{1}{2}\right)^2=\pi\quad\Rightarrow\quad\Gamma\frac{1}{2}=\sqrt{\pi}$$

Ans.

Example 15. *Show that* $\displaystyle\int_0^{\frac{\pi}{2}}\sqrt{\cot\theta}\,d\theta=\frac{1}{2}\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}$

Solution. We know that

$$\int_0^{\frac{\pi}{2}}\sin^p\theta\cdot\cos^q\theta\,d\theta=\frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}\qquad\ldots(1)$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta = \int_0^{\frac{\pi}{2}} \frac{\cos^{1/2}\theta}{\sin^{1/2}\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^{-1/2}\theta \cos^{1/2}\theta\, d\theta$$

On applying formula (1), we have

$$\int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta = \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{2\,\Gamma\left(\frac{-\frac{1}{2}+\frac{1}{2}+2}{2}\right)} = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{2\Gamma(1)} = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$ **Proved.**

Example 16. *Using Beta and Gamma functions, evaluate* $\int_0^1 \left(\frac{x^3}{1-x^3}\right)^{\frac{1}{2}} dx$ *(GBTU 2014)*

Solution.
$$\int_0^1 \left(\frac{x^3}{1-x^3}\right)^{\frac{1}{2}} dx \quad \ldots (1)$$

Putting $x^3 = \sin^2\theta$, so that $x = \sin^{\frac{2}{3}}\theta$, $dx = \frac{2}{3}\sin^{-\frac{1}{3}}\theta\, d\theta$ in (1), we get

$$\int_0^1 \left(\frac{x^3}{1-x^3}\right)^{\frac{1}{2}} dx = \int_0^{\pi/2} \left(\frac{\sin^2\theta}{1-\sin^2\theta}\right)^{\frac{1}{2}} \frac{2}{3}\sin^{-\frac{1}{3}}\theta \cos\theta\, d\theta$$

$$= \frac{2}{3}\int_0^{\pi/2} \left(\frac{\sin\theta}{\cos\theta}\right) \sin^{-\frac{1}{3}}\theta \cos\theta\, d\theta = \frac{2}{3}\int_0^{\pi/2} \sin^{\frac{2}{3}}\theta\, d\theta$$

$$= \frac{2}{3}\,\frac{\Gamma\left(\frac{\frac{2}{3}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{\frac{2}{3}+1+1}{2}\right)} = \frac{2}{3}\,\frac{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{4}{3}\right)} = \frac{2}{3}\,\frac{\sqrt{\pi}\,\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{4}{3}\right)}$$ **Ans.**

Example 17. *Evaluate* $\int_{-1}^{+1} (1+x)^{p-1} (1-x)^{q-1}\, dx.$

Solution. Put $x = \cos 2\theta$, then $dx = -2\sin 2\theta\, d\theta$

$$\int_{-1}^{+1} (1+x)^{p-1}(1-x)^{q-1} dx = \int_{\frac{\pi}{2}}^{0} (1+\cos 2\theta)^{p-1} (1-\cos 2\theta)^{q-1} (-2\sin 2\theta\, d\theta)$$

$$\int_{\frac{\pi}{2}}^{0} (1+2\cos^2\theta-1)^{p-1} (1-1+2\sin^2\theta)^{q-1} (-4\sin\theta\cos\theta\, d\theta)$$

$$= 4\int_0^{\frac{\pi}{2}} 2^{p-1}\cos^{2p-2}\theta\cdot 2^{q-1}\sin^{2q-2}\theta\cdot \sin\theta\cos\theta\, d\theta$$

$$= 2^{p+q}\int_0^{\pi} \sin^{2q-1}\theta \cos^{2p-1}\theta \, d\theta$$

$$= 2^{p+q}\frac{\overline{\left|\frac{2q}{2}\right.}\overline{\left|\frac{2p}{2}\right.}}{2\overline{\left|\frac{2p+2q}{2}\right.}} = 2^{p+q-1}\frac{\overline{|p}\,\overline{|q}}{\overline{|p+q}}$$ **Ans.**

Example 18. *Show that* $\overline{|n}\,\overline{|1-n} = \dfrac{\pi}{\sin n\pi}$ $(0 < n < 1)$

Solution. We know that

$$\beta(m,n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}}dx$$ [From (1), Art 7.7]

$$\frac{\overline{|m}\,\overline{|n}}{\overline{|m+n}} = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}}dx$$

Putting $m + n = 1$ or $m = 1 - n$, we get

$$\frac{\overline{|1-n}\,\overline{|n}}{\overline{|1}} = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^1}dx$$

$$\overline{|1-n}\,\overline{|n} = \int_0^{\infty} \frac{x^{n-1}}{1+x}dx \qquad \left[\int_0^{\infty} \frac{x^{n-1}}{1+x}dx = \frac{\pi}{\sin n\pi}\right]$$

$$\Rightarrow \qquad \overline{|n}\,\overline{|1-n} = \frac{\pi}{\sin n\pi}$$ **Proved.**

Example 19. *Assuming* $\overline{|n}\,\overline{|1-n} = \pi \operatorname{cosec} n\pi$, $0 < n < 1$, *show that*

$$\int_0^{\infty} \frac{x^{p-1}}{1+x}dx = \left(\frac{\pi}{\sin p\pi}\right), \quad 0 < p < 1$$ (U.P., I Semester, Dec 2009)

Solution. Here, we have $\pi \operatorname{cosec} n\pi = \overline{|n}\,\overline{|1-n}$

$$\Rightarrow \qquad \frac{\pi}{\sin n\pi} = \overline{|n}\,\overline{|1-n}$$

We know that $$\int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}}dx = \frac{\overline{|m}\,\overline{|n}}{\overline{|m+n}} = B(m,n) \qquad \ldots (1)$$

Putting $m + n = 1$ so that $m = 1 - n$ in (1), we get

$$\int_0^{\infty} \frac{x^{n-1}}{1+x}dx = \frac{\overline{|1-n}\,\overline{|n}}{\overline{|1}}$$ (From example 18)

$$\overline{|1-p} = \int_0^{\infty} \frac{x^{p-1}}{1+x}dx$$

$$\int_0^{\infty} \frac{x^{p-1}}{1+x}dx = \frac{\pi}{\sin px}$$ **Proved.**

Example 20. *Using Beta and Gamma functions, evaluate* $\int_0^\infty \frac{dx}{1+x^4}$. (GBTU. Dec. 2012)

Solution. Let $$I = \int_0^\infty \frac{dx}{1+x^4}$$

Putting $x^4 = y$ so that $x = y^{\frac{1}{4}}$ and $dx = \frac{1}{4} y^{-\frac{3}{4}} dy$, we get

$$I = \int_0^\infty \frac{\frac{1}{4} y^{-\frac{3}{4}}}{1+y} dy = \frac{1}{4}\int_0^\infty \frac{y^{\frac{1}{4}-1}}{1+y} dy \qquad \left[\int_0^\infty \frac{x^{n-1}}{1+x} dx = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{1}{4} \frac{\pi}{\sin\left(\frac{\pi}{4}\right)} = \frac{\pi\sqrt{2}}{4}$$ **Ans.**

Example 21. *Prove that* $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}$

Solution. Putting $n = \frac{1}{4}$ in result of example 19, we obtain

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) = \frac{\pi}{\sin\frac{\pi}{4}}$$

$$\Rightarrow \quad \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi}{\left(\frac{1}{\sqrt{2}}\right)} \Rightarrow \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}$$ **Proved.**

Example 22. *Evaluate* $\int_0^1 \frac{dx}{(1-x^n)^{\frac{1}{n}}}$

Solution. Let $x^n = \sin^2\theta$ or $x = \sin^{2/n}\theta$

So that $$dx = \frac{2}{n}\sin^{2/n-1}\theta \cos\theta \, d\theta$$

$$\int_0^1 \frac{dx}{(1-x^n)^{\frac{1}{n}}} = \int_0^{\frac{\pi}{2}} \frac{\frac{2}{n}\sin^{2/n-1}\theta \cos\theta}{(1-\sin^2\theta)^{1/n}} d\theta = \frac{2}{n}\int_0^{\frac{\pi}{2}} \frac{\sin^{2/n-1}\theta\cos\theta}{(\cos^2\theta)^{1/n}} d\theta$$

$$= \frac{2}{n}\int_0^{\frac{\pi}{2}} \sin^{2/n-1}\theta \cos^{1-2/n}\theta \, d\theta$$

$$= \frac{2}{n} \frac{\Gamma\left(\frac{\frac{2}{n}-1+1}{2}\right)\Gamma\left(\frac{1-\frac{2}{n}+1}{2}\right)}{2\Gamma\left(\frac{\frac{2}{n}-1+1+2-\frac{2}{n}}{2}\right)} = \frac{1}{n}\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{n-1}{n}\right)}{\Gamma 1} \qquad \left(\Gamma\frac{1}{n}\Gamma\left(1-\frac{1}{n}\right) = \frac{\pi}{\sin\frac{\pi}{n}}\right)$$

$$= \frac{\pi}{n\sin\frac{\pi}{n}}$$ **Ans.**

Example 23. *Show that* $\int_0^a \frac{dx}{\sqrt[n]{(a^n - x^n)}} = \frac{\pi}{n} cosec\left(\frac{\pi}{n}\right)$, *where n > 1.*

(*M.U. II Semester 2009*)

Solution. Let $x^n = a^n \sin^2\theta \quad \Rightarrow \quad x = a \sin^{\frac{2}{n}} \theta$

So that $\quad dx = \frac{2a}{n} \sin^{\frac{2}{n}-1} \theta \cos\theta \, d\theta$

$$\int_0^a \frac{dx}{\sqrt[n]{a^n - x^n}} = \int_0^{\frac{\pi}{2}} \frac{a \times \frac{2}{n} \sin^{\frac{2}{n}-1} \theta \cos\theta}{(a^n - a^n \sin^2\theta)^{\frac{1}{n}}} d\theta = \frac{2}{n} \int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{2}{n}-1} \theta \cos\theta}{\cos^{\frac{2}{n}} \theta} d\theta$$

$$= \frac{2}{n} \int_0^{\frac{\pi}{2}} \sin^{\frac{2}{n}-1} \theta \cos^{1-\frac{2}{n}} \theta \, d\theta$$

$$= \frac{2}{n} \frac{\left|\frac{\frac{2}{n}-1+1}{2}\right. \left|\frac{1-\frac{2}{n}+1}{2}\right.}{2\left|\frac{\frac{2}{n}-1+1+2-\frac{2}{n}}{2}\right.} = \frac{1}{n} \frac{\left|\frac{1}{n}\right. \left|\frac{n-1}{n}\right.}{\left|1\right.} \qquad \left[\left|\frac{1}{n}\right. \left|1-\frac{1}{n}\right. = \frac{\pi}{\sin\frac{\pi}{n}}\right]$$

$$= \frac{\pi}{n \sin\frac{\pi}{n}} = \frac{\pi}{n} \text{cosec}\left(\frac{\pi}{n}\right)$$

Proved.

Example 24. *Show that* $\int_0^{\frac{\pi}{2}} tan^P\theta \, d\theta = \frac{\pi}{2} sec \frac{P\pi}{2}$ *and indicate the restriction on the values of P.*

Solution. $\int_0^{\frac{\pi}{2}} \tan^P \theta \, d\theta = \int_0^{\frac{\pi}{2}} \sin^P \theta \cos^{-P} \theta \, d\theta$

$$= \frac{\left|\frac{P+1}{2}\right. \left|\frac{-P+1}{2}\right.}{2\left|\frac{P+1-P+1}{2}\right.} \qquad \begin{bmatrix} 1-P>0 \\ 1>P \end{bmatrix}$$

$$= \frac{\left|\frac{P+1}{2}\right. \left|\frac{-P+1}{2}\right.}{2\left|1\right.} \qquad \begin{bmatrix} 1+P>0 \\ P>-1 \end{bmatrix}$$

$$= \frac{1}{2} \left|\frac{P+1}{2}\right. \left|\frac{-P+1}{2}\right. \qquad [\therefore 1 > P > -1]$$

$$= \frac{1}{2} \frac{\pi}{\sin\frac{P+1}{2}\pi} = \frac{1}{2} \frac{\pi}{\cos\frac{P\pi}{2}} = \frac{\pi}{2} \sec\frac{P\pi}{2}$$

Proved.

7.10 DUPLICATION FORMULA

$$\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m), \text{ where } m \text{ is positive.}$$

Hence show that $\beta(m,m)=2^{1-2m}\beta\left(m,\frac{1}{2}\right)$

Proof. We know that

$$\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}=\int_0^{\frac{\pi}{2}}\sin^p\theta\cdot\cos^q\theta\,d\theta$$

Putting $q=p$, we get

$$\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}{2\Gamma(P+1)}=\int_0^{\frac{\pi}{2}}\sin^p\theta\cdot\cos^p\theta\,d\theta=\int_0^{\frac{\pi}{2}}(\sin\theta\cos\theta)^p\,d\theta$$

$$\int_0^{\frac{\pi}{2}}\frac{1}{2^p}(2\sin\theta\cos\theta)^p\,d\theta=\frac{1}{2^p}\int_0^{\frac{\pi}{2}}(\sin 2\theta)^p\,d\theta$$

$$\left[\text{Putting } 2\theta=t\Rightarrow d\theta=\frac{dt}{2}\right]$$

$$=\frac{1}{2^p}\int_0^{\pi}\sin^p t\frac{dt}{2}$$

$$=\frac{1}{2^p}\cdot\frac{1}{2}\cdot 2\int_0^{\frac{\pi}{2}}\sin^p t\,dt=\frac{1}{2^p}\int_0^{\frac{\pi}{2}}\sin^p t\,\cos^0 t\,dt$$

$$=\frac{1}{2^p}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{p+2}{2}\right)}$$

$$\Rightarrow\quad\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}{2\Gamma(p+1)}=\frac{1}{2^p}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{p+2}{2}\right)}$$

$$\Rightarrow\quad\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma(p+1)}=\frac{1}{2^p}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)}$$

$$\Rightarrow\quad\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma(p+1)}=\frac{1}{2^p}\frac{\sqrt{\pi}}{\Gamma\left(\frac{p+2}{2}\right)}$$

Take $\frac{p+1}{2}=m \quad\Rightarrow\quad p=2m-1$

... (1)

$$\Rightarrow \quad \frac{\overline{|m}}{\overline{|2m}} = \frac{1}{2^{2m-1}} \frac{\sqrt{\pi}}{\overline{\left|\frac{2m+1}{2}\right.}}$$

Proved.

$$\overline{|m}\,\overline{\left|m+\frac{1}{2}\right.} = \frac{\sqrt{\pi}}{2^{2m-1}}\overline{|2m}$$

Multiplying both sides of (1) by $\overline{|m}$, we have

$$\frac{\overline{|m}\,\overline{|m}}{\overline{|2m}} = 2^{1-2m}\frac{\overline{\left|\frac{1}{2}\right.}\,\overline{|m}}{\overline{\left|m+\frac{1}{2}\right.}}$$

$$\boxed{\beta(m,m) = 2^{1-2m}\beta\left(m,\frac{1}{2}\right)}$$

Proved.

Example 25. *For α β function, show that β (p, q) = β (p + 1, q) + β (p, q + 1)* (U.P., I[st] Semester, Dec 2008)

Solution. $\beta(p+1, q) + \beta(p, q+1)$

$$= \int_0^1 x^p (1-x)^{q-1} dx + \int_0^1 x^{p-1} (1-x)^q dx$$

$$= \int_0^1 x^{p-1} (1-x)^{q-1} [x + 1 - x] dx \quad \text{(Taking common)}$$

$$= \int_0^1 x^{p-1} (1-x)^{q-1} dx$$

$$= \beta(p, q)$$

Proved.

Example 26. *Prove that* $\beta(m,m)\times\beta\left(m+\frac{1}{2}, m+\frac{1}{2}\right) = \frac{\pi}{m}2^{1-4m}$ (M.T.U II, Semester, 2008)

Solution.

$$\text{L.H.S.} = \beta(m,m)\times\beta\left(m+\frac{1}{2}, m+\frac{1}{2}\right)$$

$$\frac{\overline{|m}\,\overline{|m}}{\overline{|2m}} \times \frac{\overline{\left|m+\frac{1}{2}\right.}\,\overline{\left|m+\frac{1}{2}\right.}}{\overline{|2m+1}}$$

$$= \frac{(\overline{|m})^2}{\overline{|2m}} \times \frac{\left(\overline{\left|m+\frac{1}{2}\right.}\right)^2}{2m\overline{|2m}} \qquad [\because \overline{|2m+1} = 2m\overline{|2m}]$$

$$\left(\frac{\overline{|m}\,\overline{\left|m+\frac{1}{2}\right.}}{\overline{|2m}}\right)^2 \cdot \frac{1}{2m}$$

$$= \left(\frac{\sqrt{\pi}}{2^{2m-1}}\right)^2 \frac{1}{2m}$$

$$= \frac{\pi}{2^{4m-2}} \cdot \frac{1}{2m}$$

$$= \frac{\pi}{m} \cdot 2^{1-4m}$$

$$= \text{R.H.S.}$$

$$\left[\begin{array}{l} \text{By Duplication formula} \\ 2^{2m-1} \cdot \Gamma(m) \cdot \Gamma\left(m+\frac{1}{2}\right) = \sqrt{\pi} \cdot \Gamma(2m) \\ \Rightarrow \quad \dfrac{\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(2m)} = \dfrac{\sqrt{\pi}}{2^{2m-1}} \end{array}\right]$$

Proved.

7.11 TO SHOW THAT

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\ldots\Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{\left(\frac{n-1}{2}\right)}}{n^{1/2}}$$

where n is a positive integer than one.

Proof. Let

$$P = \Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\ldots\Gamma\left(\frac{n-2}{n}\right)\Gamma\left(\frac{n-1}{n}\right)$$

$$= \Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\ldots\Gamma\left(1-\frac{2}{n}\right)\Gamma\left(1-\frac{1}{n}\right) \qquad \ldots (1)$$

Writing the value of P in the reverse order, we have

$$P = \Gamma\left(1-\frac{1}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\ldots\ldots\Gamma\left(\frac{3}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{1}{n}\right) \qquad \ldots (2)$$

Multiplying (1) and (2), we get

$$P^2 = \left(\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)\right)\left(\Gamma\left(\frac{2}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\right)\ldots\ldots\left(\Gamma\left(1-\frac{2}{n}\right)\Gamma\left(\frac{2}{n}\right)\right)\left(\Gamma\left(1-\frac{1}{n}\right)\Gamma\left(\frac{1}{n}\right)\right)$$

$$P^2 = \frac{\pi}{\sin\left(\frac{\pi}{n}\right)} \cdot \frac{\pi}{\sin\left(\frac{2\pi}{n}\right)} \cdot \frac{\pi}{\sin\left(\frac{3\pi}{n}\right)} \ldots\ldots \frac{\pi}{\sin\frac{(n-1)\pi}{n}} \quad \left[\because \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right]$$

$$\Rightarrow \quad P^2 = \frac{\pi^{n-1}}{\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right)\ldots\ldots\sin\left\{\frac{(n-1)\pi}{n}\right\}} \qquad \ldots (3)$$

But from Trigonometry, we know that

$$\frac{\sin n\theta}{\sin\theta} = 2^{n-1}\sin\left(\theta+\frac{\pi}{n}\right)\sin\left(\theta+\frac{2\pi}{n}\right)\ldots\ldots\sin\left(\theta+\frac{(n-1)\pi}{n}\right) \qquad \ldots (4)$$

Take Limit as $\theta \to 0$,

$$\underset{\theta\to 0}{\text{Lt}} \frac{\sin n\theta}{\sin\theta} = \underset{\theta\to 0}{\text{Lt}}\left(n \cdot \frac{\sin n\theta}{\sin\theta} \cdot \frac{\theta}{\sin\theta}\right) = n$$

On putting this limit in (4), we get

$$n = 2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\sin\frac{3\pi}{n}\ldots\sin\frac{(n-1)\pi}{n}$$ [From (4)]

$$\Rightarrow \quad \sin\frac{\pi}{n}\cdot\sin\frac{2\pi}{n}\cdot\sin\frac{3\pi}{n}\ldots\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$$

Substituting this in equation (3), we obtain

$$P^2 = \frac{\pi^{n-1}}{\left(\frac{n}{2^{n-1}}\right)} = \frac{(2\pi)^{n-1}}{n}$$

$$\therefore \quad P = \frac{(2\pi)^{(n-1)/2}}{n^{1/2}}$$

$$\Rightarrow \quad \boxed{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\ldots\Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{(n-1)/2}}{n^{1/2}}}$$

7.12 TO SHOW THAT

(i) $\int_0^\infty e^{-ax}x^{n-1}\cos bx\,dx = \frac{\Gamma(n)\cos n\theta}{(a^2+b^2)^{n/2}}$

(ii) $\int_0^\infty e^{-ax}x^{n-1}\sin bx\,dx = \frac{\Gamma(n)\sin n\theta}{(a^2+b^2)^{n/2}}$ where $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

Proof. We know that $\int_0^\infty e^{-ax}\cdot x^{n-1}\,dx = \frac{\Gamma(n)}{a^n}$, where a, n are positive.

Put $ax = z$ so that $\quad dx = \frac{dz}{a}$

$$\therefore \quad \int_0^\infty e^{-ax}x^{n-1}\,dx = \int_0^\infty e^{-z}\left(\frac{z}{a}\right)^{n-1}\cdot\frac{dz}{a} = \frac{1}{a^n}\int_0^\infty e^{-z}z^{n-1}\,dz = \frac{\Gamma(n)}{a^n}$$

Replacing a by $(a+ib)$, we have

$$\int_0^\infty e^{-(a+ib)x}x^{n-1}\,dx = \frac{\Gamma n}{(a+ib)^n} \quad \ldots(1)$$

Now

$$e^{-(a+ib)x} = e^{-ax}\cdot e^{-ibx} = e^{-ax}(\cos bx - i\sin bx)$$

Putting the value of $e^{-(a+ib)x}$ in (1), we get

$$\int_0^\infty e^{-ax}(\cos bx - i\sin bx)x^{n-1}\,dx = \frac{\Gamma n}{(a+ib)^n} \quad \ldots(2)$$

Putting $a = r\cos\theta$ and $b = r\sin\theta$ so that $r^2 = a^2 + b^2$ and $\theta = \tan^{-1}\frac{b}{a}$

$(a+ib)^n = (r\cos\theta + ir\sin\theta)^n = r^n(\cos\theta + i\sin\theta)^n$

$(a+ib)^n = r^n(\cos n\theta + i\sin n\theta)$ [De Moivre's Theorem]

Putting the value of $(a+ib)^n$ in (2), we have

$$\int_0^\infty e^{-ax}(\cos bx - i\sin bx)x^{n-1}dx = \frac{\Gamma n}{r^n(\cos n\theta + i\sin n\theta)}$$

$$= \frac{\Gamma n}{r^n}(\cos n\theta + i\sin n\theta)^{-1}$$

$$= \frac{\Gamma n}{r^n}(\cos n\theta - i\sin n\theta)$$

Now, equating real and imaginary parts on the two sides, we get

(*i*) $\int_0^\infty e^{-ax}x^{n-1}\cos bx\, dx = \frac{\Gamma n}{r^n}\cos n\theta$ and

(*ii*) $\int_0^\infty e^{-ax}x^{n-1}\sin bx\, dx = \frac{\Gamma n}{r^n}\sin n\theta$

Where $r^2 = a^2 + b^2$ and $\theta = \tan^{-1}\frac{b}{a}$.

Example 27. *Evaluate:*

(*i*) $\int_0^\infty \cos x^2 dx$ (*ii*) $\int_{-\infty}^\infty \cos\frac{\pi x^2}{2}dx.$

Solution. (*i*) We know that

$$\int_0^\infty e^{-ax}\cdot x^{n-1}\cos bx\, dx = \frac{\Gamma n\cos n\theta}{(a^2+b^2)^{n/2}},\text{ where } \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Put $a = 0$, $\int_0^\infty x^{n-1}\cos bx\, dx = \frac{\Gamma n}{b^n}\cos\frac{n\pi}{2}$ $\left[\begin{aligned}\theta &= \tan^{-1}\left(\frac{b}{0}\right)\\ &= \tan^{-1}(\infty) = \frac{\pi}{2}\end{aligned}\right]$

Put $x^n = z$ so that $x^{n-1}dx = \frac{dz}{n}$ and $x = z^{1/n}$

then, $$\int_0^\infty \cos bz^{1/n}dz = \frac{n\Gamma n}{b^n}\cos\frac{n\pi}{2}$$

$$\Rightarrow \int_0^\infty \cos(bx^{1/n})\,dx = \frac{\Gamma(n+1)}{b^n}\cos\frac{n\pi}{2} \quad \ldots (1)$$

Here $b = 1$, $n = \frac{1}{2}$

$$\therefore \int_0^\infty \cos x^2 dx = \Gamma\left(\frac{3}{2}\right)\cos\frac{\pi}{4} = \frac{\sqrt{\pi}}{2}\cdot\frac{1}{\sqrt{2}} = \frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{2}} \quad \textbf{Ans.}$$

(*ii*) $$I = \int_{-\infty}^\infty \cos\frac{\pi x^2}{2}dx = 2\int_0^\infty \cos\frac{\pi x^2}{2}dx \quad \ldots (2)$$

Putting $b = \frac{\pi}{2}$ and $n = \frac{1}{2}$ in equation (1), we get

$$\int_0^\infty \cos\left(\frac{\pi}{2}x^2\right)dx = \frac{\Gamma\left(\frac{3}{2}\right)}{\left(\frac{\pi}{2}\right)^{1/2}}\cos\frac{\pi}{4}$$

$\therefore$ From (2),

$$I = 2\frac{\Gamma\left(\frac{3}{2}\right)}{\left(\frac{\pi}{2}\right)^{1/2}}\cos\frac{\pi}{4} = 2\cdot\frac{1}{2}\sqrt{\pi}\cdot\frac{\sqrt{2}}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{2}} = 1.$$

Ans.

Example 28. *Evaluate:* $\int_0^1 log\,\Gamma(x)\,dx$

Solution. Let

$$I = \int_0^1 \log\Gamma(x)\,dx \qquad \ldots (1)$$

$$= \int_0^1 \log\Gamma(1-x)\,dx \qquad \left[\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right] \qquad \ldots (2)$$

Adding (1) and (2), we get

$$2I = \int_0^1 (\log\Gamma(x) + \log\Gamma(1-x))\,dx$$

$$= \int_0^1 \log(\Gamma(x)\Gamma(1-x))\,dx = \int_0^1 \log\left(\frac{\pi}{\sin\pi x}\right)dx \qquad \left[\Gamma x\,\Gamma 1-x = \frac{\pi}{\sin\pi x}\right]$$

$$= \int_0^1 (\log\pi - \log\ \sin\pi x)\,dx = \int_0^1 \log\pi\,dx - \int_0^1 \log\ \sin\pi x\,dx \qquad \ldots (3)$$

$$= I_1 - I_2$$

where

$$I_1 = \int_0^1 \log\pi\,dx = \log\pi$$

$$I_2 = \int_0^1 \log\sin\pi x\,dx \qquad \left[\text{Put } \pi x = t \Rightarrow dx = \frac{1}{\pi}dt\right]$$

$$= \int_0^\pi \log\sin t\left(\frac{dt}{\pi}\right)$$

$$\frac{1}{\pi}\cdot 2\int_0^{\pi/2}\log\sin t\,dt = \frac{2}{\pi}\left(-\frac{\pi}{2}\log 2\right) = -\log 2$$

From (3),

$$2I = \log\pi + \log 2 = \log 2\pi$$

$$I = \frac{1}{2}\log 2\pi.$$

Ans.

Example 29. *Prove that* $\int_0^\infty x^2 e^{-x^4}dx \times \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}dx = \frac{\pi}{4\sqrt{2}}$ *(M.U., II Semester, 2008)*

Solution. Here, we have

$$\int_0^\infty x^2 e^{-x^4}dx \qquad \ldots (1)$$

Putting $x^4 = t \Rightarrow x = t^{\frac{1}{4}}$, $dx = \frac{1}{4} t^{-\frac{3}{4}} dt$ in (1), we get

$$\int_0^\infty t^{\frac{1}{2}} e^{-t}\left(\frac{1}{4} t^{-\frac{3}{4}} dt\right) = \frac{1}{4}\int_0^\infty e^{-t} t^{-\frac{1}{4}} dt$$

$$= \frac{1}{4}\int_0^\infty e^{-t} t^{\frac{3}{4}-1} dt$$

$$= \frac{1}{4}\Gamma\left(\frac{3}{4}\right) \qquad \ldots (2)$$

Now, we calculate the value of $\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}} dx$ ... (3)

Putting $x^2 = t$, $x = t^{\frac{1}{2}} \Rightarrow dx = \frac{1}{2} t^{-\frac{1}{2}} dt$ in (2), we get

$$\int_0^\infty e^{-t} t^{-\frac{1}{4}}\left(\frac{1}{2} t^{-\frac{1}{2}} dt\right) = \frac{1}{2}\int_0^\infty e^{-t} t^{-\frac{3}{4}} dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t} t^{\frac{1}{4}-1} dt$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{4}\right) \qquad \ldots (4)$$

From (3) and (4), we get

$$\int_0^\infty x^2 e^{-x^4} dx \cdot \int_0^\infty \frac{e^{-x^2}}{\Gamma x} dx = \left(\frac{1}{4}\Gamma\left(\frac{3}{4}\right)\right)\left(\frac{1}{2}\Gamma\left(\frac{1}{4}\right)\right) = \frac{1}{8}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)}$$

Duplication formula

$$2^{2m-1}\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2m)$$

Putting $m = \frac{1}{4}$, we get

$$2^{-\frac{1}{2}}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{\pi}\,\Gamma\left(\frac{1}{2}\right)$$

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{\pi}\sqrt{\pi}(2)^{\frac{1}{2}} = \pi\sqrt{2}$$

$$= \frac{1}{8}\left(\pi\sqrt{2}\right) = \frac{\pi}{4\sqrt{2}}$$

Proved.

Example 30. *Evaluate* $\iint_A \frac{dx\,dy}{\sqrt{xy}}$ *using the substitutions*

$$x = \frac{u}{1+v^2}, \quad y = \frac{uv}{1+v^2}$$

where A is bounded by $x^2 + y^2 - x = 0,\ y = 0,\ y > 0.$

Solution. Here $\sqrt{xy} = \sqrt{\left(\frac{u}{1+v^2}\right)\left(\frac{uv}{1+v^2}\right)} = \frac{u\sqrt{v}}{1+v^2}$

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} du\,dv = \begin{vmatrix} \frac{1}{1+v^2} & -\frac{2uv}{(1+v^2)^2} \\ \frac{v}{1+v^2} & \frac{u(1-v^2)}{(1+v^2)^2} \end{vmatrix} du\,dv$$

$$= \left[\frac{u(1-v^2)}{(1+v^2)^3} + \frac{2uv^2}{(1+v^2)^3}\right] du\,dv = \left[\frac{u - uv^2 + 2uv^2}{(1+v^2)^3}\right] du\,dv$$

$$= \frac{u(1+v^2)}{(1+v^2)^3} du\,dv = \frac{u}{(1+v^2)^2} du\,dv$$

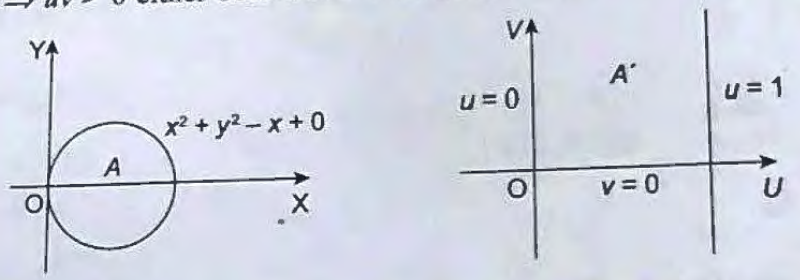
Also the circle $x^2 + y^2 - x = 0$ is transformed into

$$\frac{u^2}{(1+v^2)^2} + \frac{u^2v^2}{(1+v^2)^2} - \frac{u}{1+v^2} = 0 \Rightarrow \frac{u^2(1+v^2)}{(1+v^2)^2} - \frac{u}{1+v^2} = 0$$

$$\Rightarrow \qquad \frac{u^2}{1+v^2} - \frac{u}{1+v^2} = 0 \Rightarrow u^2 - u = 0 \Rightarrow u(u-1) = 0 \Rightarrow u = 0,\ u = 1$$

Further $\qquad y = 0 \Rightarrow \frac{uv}{1+v^2} = 0 \Rightarrow u = 0,\ v = 0$

and $y > 0 \Rightarrow uv > 0$ either both u and v are positive or both negative.



The area A, *i.e.*, $x^2 + y^2 - x = 0$ is transformed into A' bounded by $u = 0$, $v = 0$ and $u = 1$ and $v = \infty$.

$$\iint \frac{dx\,dy}{\sqrt{x}} = \int_0^1\int_0^\infty \frac{\frac{u}{(1+v^2)^2}}{\frac{u\sqrt{v}}{1+v^2}} du\,dv = \int_0^1\int_0^\infty \frac{1}{\sqrt{v}(1+v^2)} dv\,du$$

On putting $v = \tan\theta$, $dv = \sec^2\theta\, d\theta$

$$= \int_0^1\int_0^{\frac{\pi}{2}} \frac{\sec^2\theta}{\sqrt{\tan\theta}(1+\tan^2\theta)} d\theta\,du = \int_0^1 du \int_0^{\frac{\pi}{2}} \sqrt{\frac{\cos\theta}{\sin\theta}} d\theta = \int_0^1 du \int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{2}}\theta \cos^{\frac{1}{2}}\theta\, d\theta$$

Duplication formula $\Gamma m\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma 2m$

$$= \int_0^1 du \frac{\Gamma\left(-\frac{1}{2}+1\right)\Gamma\left(\frac{1}{2}+1\right)}{\Gamma 2\;\Gamma 2} = \frac{1}{2}\int_0^1 du\, \Gamma\frac{1}{4}\Gamma\frac{3}{4} = \frac{1}{2}\int_0^1 du\left[\frac{\sqrt{\pi}}{2^{-\frac{1}{2}}}\Gamma\frac{1}{2}\right]$$

$$= \frac{1}{2}\int_0^1 du\,\Gamma 2\sqrt{\pi}\cdot\sqrt{\pi} = \frac{\pi}{\Gamma 2}[u]_0^1 = \frac{\pi}{\Gamma 2}$$ **Ans.**

Example 31. *Prove that* $\iint_D x^{l-1}y^{m-1}dx\,dy = \frac{\Gamma l\,\Gamma m}{\Gamma(l+m+1)}h^{l+m}$ *where D is the domain* $x \geq 0,\ y \geq 0$ *and* $x + y \leq h$.

Solution. Putting $x = Xh$ and $y = Yh$, $dx\,dy = h^2\,dX\,dY$

$$\iint_D x^{l-1}y^{m-1}dx\,dy = \iint_{D'}(Xh)^{l-1}(Yh)^{m-1}h^2 dX\,dY$$

where D' is the domain

$X \geq 0,\ \ Y \geq 0,\ \ X + Y \leq 1$

$$= h^{l+m}\int_0^1\int_0^{1-X} X^{l-1}Y^{m-1}dX\,dY = h^{l+m}\int_0^1 X^{l-1}dX\int_0^{1-X}Y^{m-1}dY$$

$$= h^{l+m}\int_0^1 X^{l-1}dX\left[\frac{Y^m}{m}\right]_0^{1-X} = \frac{h^{l+m}}{m}\int_0^1 X^{l-1}(1-X)^m\,dX$$

$$= \frac{h^{l+m}}{m}\beta(l, m+1) = \frac{h^{l+m}}{m}\frac{\Gamma l\,\Gamma(m+1)}{\Gamma(l+m+1)}$$

$$= \frac{h^{l+m}}{m}\frac{\Gamma l\; m\Gamma m}{\Gamma(l+m+1)} = h^{l+m}\frac{\Gamma l\,\Gamma m}{\Gamma(l+m+1)}$$ **Proved.**

7.13 DIRICHLET'S INTEGRAL

If l, m, n are all positive, then the triple integral

$$\iiint_V x^{l-1}y^{m-1}z^{n-1}dx\,dy\,dz = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}$$

where V is the region $x \geq 0,\ y \geq 0,\ z \geq 0$ and $0 \leq x + y + z \leq 1$.

Proof. Putting $\quad y + z \leq 1 - x = h$. Then $z \leq h - y$

$$\iiint_V x^{l-1}y^{m-1}z^{n-1}dx\,dy\,dz = \int_0^1 x^{l-1}dx\int_0^{1-x}y^{m-1}dy\int_0^{1-x-y}z^{n-1}dz$$

$$= \int_0^1 x^{l-1}dx\left[\int_0^h\int_0^{h-y}y^{m-1}z^{n-1}dy\,dz\right]$$ [Put $x = h$]

1 - x = h

$$= \frac{\Gamma m\,\Gamma n}{\Gamma(m+n+1)}\bullet\int_0^1 x^{l-1}(1-x)^{m+n}\,dx$$

$$= \frac{\Gamma m \, \Gamma n}{\Gamma m+n+1} \beta(l, m+n+1)$$

$$= \frac{\Gamma m \, \Gamma n}{\Gamma m+n+1} \frac{\Gamma l \, \Gamma m+n+1}{\Gamma l+m+n+1}$$

$$\iiint_V x^{l-1} y^{m-1} z^{n-1} dx\, dy\, dz = \frac{\Gamma l \, \Gamma m \, \Gamma n}{\Gamma l+m+n+1}$$

Note: $\iiint_V x^{l-1} y^{m-1} z^{n-1} dx\, dy\, dz = \frac{\Gamma l \, \Gamma m \, \Gamma n}{\Gamma l+m+n+1} h^{l+m+n}$ where V is the domain, $x \geq 0$, $y \geq 0$, $z \geq 0$ and $x+y+z \leq h$.

Corollary: Dirichlet's theorem for n variables, the theorem states that

$$\iiint \ldots \int x_1^{l_1-1} x_2^{l_2-1} \ldots x_n^{l_n-1} dx_1\, dx_2\, dx_3 \ldots dx_n = \frac{\Gamma l_1 \, \Gamma l_2 \, \Gamma l_3 \ldots \Gamma l_n}{\Gamma 1+l_1+l_2+\ldots+l_n} h^{l_1+l_2+\ldots+l_n}$$

Example 32. *Show that* $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}} dx = \frac{\beta(m,n)}{a^n(1+a)^m}$

Solution. Put $\frac{x}{a+x} = \frac{t}{a+1}$

$$(a+1)x = t(a+x) \Rightarrow x = \frac{at}{a+1-t}$$

$$dx = \frac{(a+1-t)a\,dt - at(-dt)}{(a+1-t)^2}$$

$$\frac{(a^2+a-at+at)}{(a+1-t)^2} dt = \frac{a(a+1)}{(a+1-t)^2} dt$$

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}} dx = \int_0^1 \frac{\left(\frac{at}{a+1-t}\right)^{m-1} \cdot \left(1-\frac{at}{a+1-t}\right)^{n-1}}{\left(a+\frac{at}{a+1-t}\right)^{m+n}} \frac{a(a+1)}{(a+1-t)^2} dt$$

$$= \int_0^1 \frac{(at)^{m-1}(a+1-t-at)^{n-1}}{(a^2+a-at+at)^{m+n}} a(a+1)\, dt$$

$$= \int_0^1 \frac{a^{m-1} t^{m-1} (a+1)^{n-1} (1-t)^{n-1}}{a^{m+n}(a+1)^{m+n}} a(a+1)\, dt\,.$$

$$= \frac{1}{a^n(a+1)^m} \int_0^1 t^{m-1}(1-t)^{n-1} dt$$

$$= \frac{1}{a^n(a+1)^m} \beta(m,n)$$

Example 33. *Prove that* $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}dx = \frac{1}{5005}$. (M.U. II Semester, 2008)

Solution. Let
$$I = \int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}dx$$

$\Rightarrow$
$$I = \int_0^\infty \frac{x^4}{(1+x)^{15}}dx + \int_0^\infty \frac{x^9}{(1+x)^{15}}dx$$
$$= I_1 + I_2 \qquad \dots (1)$$

Now, put
$$x = \frac{t}{1-t}, \text{ when } x = 0, t = 0; \text{ when } x = \infty, t = 1$$
$$1 + x = 1 + \frac{t}{1-t} = \frac{1}{1-t} \Rightarrow dx = \frac{dt}{(1-t)^2}$$

$\therefore$
$$I_1 = \int_0^1 \left(\frac{t}{1-t}\right)^4 \cdot (1-t)^{15} \cdot \frac{1}{(1-t)^2}dt$$
$$= \int_0^1 t^4 (1+t^9)\,dt = \beta(5,10) \qquad \dots (2)$$

and
$$I_2 = \int_0^1 \left(\frac{t}{1-t}\right)^9 \cdot (1-t)^{15} \cdot \frac{dt}{(1-t)^2}$$
$$= \int_0^1 t^9 (1-t^4)\,dt = \beta(10,5) \qquad \dots (3)$$

$\therefore$
$$I = I_1 + I_2$$
$$= \beta(5,10) + \beta(10,5) \qquad \text{[using (2) and (3)]}$$
$$= \beta(5,10) + \beta(5,10) \qquad [\because \beta(m,n) = \beta(n,m)]$$
$$= 2\beta(5,10)$$
$$= \frac{2\sqrt{5}\sqrt{10}}{\sqrt{15}} = \frac{2\cdot 4!\cdot 9!}{14!}$$
$$= \frac{2\times4\times3\times2\times1\times9!}{14\times13\times12\times10\times9!} = \frac{1}{7\times13\times11\times5} = \frac{1}{5005} \qquad \textbf{Proved.}$$

Example 34. *Find the value of*
$$\int_0^1 \frac{x^3 - 2x^4 + x^5}{(1+x)^7}dx$$
(M.U., II Semester, 2008)

Solution. Let
$$I = \int_0^1 \frac{x^3 - 2x^4 + x^5}{(1+x)^7}dx$$

$\Rightarrow$
$$I = \int_0^1 \frac{x^3}{(1+x)^7}dx - 2\int_0^1 \frac{x^4}{(1+x)^7}dx + \int_0^1 \frac{x^5}{(1+x)^7}dx \qquad \dots (1)$$

$\Rightarrow$
$$I = I_1 - 2I_2 + I_3$$

Now put $x = \frac{t}{1-t}$ so that $1 + x = 1 + \frac{t}{1-t} \Rightarrow 1 + x = \frac{1}{1-t} \Rightarrow dx = \frac{dt}{(1-t)^2}$

$$\therefore \quad I_1 = \int_0^1 \left(\frac{t}{1-t}\right)^3 \cdot (1-t)^7 \cdot \frac{1}{(1-t)^2}\, dt$$

$$= \int_0^1 t^3 (1-t)^2\, dt = \int_0^1 t^{4-1} (1-t)^{3-1}\, dt = \beta(4,3) \qquad \cdots (2)$$

and

$$I_2 = \int_0^1 \left(\frac{t}{1-t}\right)^4 (1-t)^7 \cdot \frac{1}{(1-t)^2}\, dt$$

$$= \int_0^1 t^4 (1-t)\, dt = \int_0^1 t^{5-1} (1-t)^{2-1}\, dt = \beta(5,2) \qquad \cdots (3)$$

Also, $I_3 = \int_0^1 \left(\frac{t}{1-t}\right)^5 (1-t)^7 \frac{dt}{(1-t)^2} = \int_0^1 t^5 (1-t)^0\, dt = \int_0^1 t^{6-1} (1-t)^{1-1}\, dt = \beta(6,1) \qquad \cdots (4)$

Putting the values of I_1, I_2 and I_3 in (1), we get

$$I = \beta(4, 3) - 2\beta(5, 2) + \beta(6, 1)$$

$$= \frac{\Gamma 4\, \Gamma 3}{\Gamma(4+3)} - 2\frac{\Gamma 5\, \Gamma 2}{\Gamma(5+2)} + \frac{\Gamma 6\, \Gamma 1}{\Gamma(6+1)}$$

$$= \frac{\Gamma 4\, \Gamma 3}{\Gamma 7} - 2\frac{\Gamma 5\, \Gamma 2}{\Gamma 7} + \frac{\Gamma 6\, \Gamma 1}{\Gamma 7}$$

$$= \frac{\Gamma 3}{4 \times 5 \times 6} - 2\frac{1}{5 \times 6} + \frac{1}{6}$$

$$= \frac{1}{60} - \frac{1}{15} + \frac{1}{6} = \frac{1 - 4 + 10}{60} = \frac{7}{60} \qquad \textbf{Ans.}$$

EXERCISE 7.2

Prove that:

1. (a) $\int_0^{\frac{\pi}{2}} \sin^2\theta \cos^4\theta\, d\theta = \frac{\pi}{32}$ (b) $\int_0^{\frac{\pi}{2}} \sin^6\theta\, d\theta = \frac{5\pi}{32}$

2. (a) $\beta(m+1, n) = \frac{m}{m+n}\beta(m,n)$ *(U.P. I Sem. Jan 2011)* (b) $\beta(m, n+1) = \frac{n}{m+n}\beta(m,n)$

3. $\int_0^1 \sqrt{x}\, \sqrt[3]{1-x^2}\, dx = \dfrac{\Gamma\frac{3}{4}\, \Gamma\frac{4}{3}}{2\, \Gamma\frac{7}{12}}$

4. $\int_0^1 (1-x^n)^{\frac{1}{2}}\, dx = \dfrac{\Gamma\frac{1}{n}\, \Gamma\frac{1}{2}}{n\, \Gamma\frac{n+2}{2n}}$

5. $\int_0^1 (1-x^{\frac{1}{n}})^m \, dx = \dfrac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$

6. $\int_1^\infty \dfrac{dx}{x^{P+1}(x-1)^q} = \beta(P+q, 1-q) \text{ if } -P < q < 1$

7. $\int_0^1 x^m (1-x^n)^P \, dx = \dfrac{1}{n} \dfrac{\Gamma\left(\frac{m+1}{2}\right) \Gamma(P+1)}{\Gamma\left(\frac{m+1}{n} + P + 1\right)}$

8. $\int_a^b (x-a)^m (b-x)^n \, dx = (b-a)^{m+n+1} \cdot \beta(m+1, n+1)$

9. $\int_3^7 \sqrt[4]{(x-3)(7-x)} \, dx = \dfrac{2\left(\Gamma\frac{1}{4}\right)^2}{3\sqrt{\pi}}$ **[Hint.** Put $x = 4t + 3$**]**

10. $\int_0^{\frac{\pi}{2}} \dfrac{d\theta}{\sqrt{1 - \frac{1}{2}\sin^2\theta}} = \dfrac{\left(\Gamma\frac{1}{4}\right)^2}{4\sqrt{\pi}}$

11. If $\int_0^\infty e^{-x} x^{n-1} \, dx = I_n$ for $n > 0$ find $\dfrac{I_{n+1}}{I_n}$ **Ans.** n

Show that:

12. $\int_0^\infty \sqrt{x} e^{-x^2} \, dx \times \int_0^\infty \dfrac{e^{-x^2}}{\sqrt{x}} \, dx = \dfrac{\pi}{2\sqrt{2}}$

13. $\int_0^1 \dfrac{x^2 \, dx}{\sqrt{1-x^4}} \int_0^1 \dfrac{dx}{\sqrt{1+x^4}} = \dfrac{\pi}{4\sqrt{2}}$

14. $\int_0^\infty x^2 e^{-x^4} \, dx \times \int_0^\infty e^{-x^4} \, dx = \dfrac{\pi}{8\sqrt{2}}$

15. $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^{5/2} x \, dx = \dfrac{8}{77}$

16. $\int_0^{\pi/2} \tan^n x \, dx = \dfrac{\pi}{2} \sec \dfrac{n\pi}{2}; -1 < n < 1.$

17. $\int_0^1 \dfrac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}} \, dx = \dfrac{\beta(m,n)}{a^n (a+b)^m}$

18. $\dfrac{\beta(p, q+1)}{q} = \dfrac{\beta(p+1, q)}{p} = \dfrac{\beta(p,q)}{p+q}$

7.14 LIOUVILLE'S EXTENSION OF DIRICHLET THEOREM

If the variables x, y, z are all positive such that $h_1 < x + y + z < h_2$, then

$$\iiint f(x+y+z)\,x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n)}\int_{h_1}^{h_2} f(u)\,u^{l+m+n-1}\,du$$

Proof. By Dirichlet Theorem, we have

$$I = \iiint X^{l-1}Y^{m-1}Z^{n-1}\,dX\,dY\,dZ = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(1+l+m+n)} \quad \ldots (1)$$

Under the condition $x + y + z \le u \Rightarrow \frac{x}{u}+\frac{y}{u}+\frac{z}{u} \le 1$

Putting $X = \frac{x}{u}$, $Y = \frac{y}{u}$ and $Z = \frac{z}{u}$ so that

$dX = \frac{dx}{u}$, $dY = \frac{dy}{u}$, $dZ = \frac{dz}{u}$ in (1), we get

$$\iiint \left(\frac{x}{u}\right)^{l-1}\left(\frac{y}{u}\right)^{m-1}\left(\frac{z}{u}\right)^{n-1}\frac{dx}{u}\frac{dy}{u}\frac{dz}{u} = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(1+l+m+n)}$$

$$\iiint x^{l-1}\,y^{m-1}\,z^{n-1}\,dx\,dy\,dz = u^{l+m+n}\,\frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(1+l+m+n)}$$

Similarly, if $x + y + z \le u + \delta u$ then

$$\iiint x^{l-1}\,y^{m-1}\,z^{n-1}\,dx\,dy\,dz = (u+\delta u)^{l+m+n}\,\frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(1+l+m+n)}$$

Hence, value of the integral I extended to all such values of the variables as make the sum of the variable lie between u and $u + \delta u$ is given by

$$I = (u+\delta u)^{l+m+n}\frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(1+l+m+n)} - u^{l+m+n}\frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(1+l+m+n)}$$

$$\iiint x^{l-1}\,y^{m-1}\,z^{n-1}\,dx\,dy\,dz = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(1+l+m+n)}\left[(u+\delta u)^{l+m+n} - u^{l+m+n}\right]$$

$$= \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(1+l+m+n)}\,u^{l+m+n}\left[\left(1+\frac{\delta}{u}\right)^{l+m+n} - 1\right]$$

$$= \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(1+l+m+n)}\,u^{l+m+n}\left[1+(l+m+n)\frac{\delta u}{u}+\ldots\ldots-1\right]$$

$$= \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(1+l+m+n)}\,u^{l+m+n}\,(l+m+n)\frac{\delta u}{u} = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(1+l+m+n)}\,u^{l+m+n-1}\,\delta u$$

Let us consider $\iiint f(x+y+z)\,x^{l-1}\,y^{m-1}\,z^{n-1}\,dx\,dy\,dz$

Under the condition $h_1 \le x + y + z \le h_2$, when $x + y + z$ lies between u and $u + \delta u$, the value of $f(x + y + z)$ can only differ from $f(u)$ by a small quantity of the same order as δu. Hence, neglecting square of δu, the part of the integral

$$\iiint f(x+y+z)\,x^{l-1}\,y^{m-1}\,z^{n-1}\,dx\,dy\,dz = \frac{\overline{|l}\,\overline{|m}\,\overline{|n}}{\overline{|l+m+n}} f(u)\,u^{l+m+n-1}\,\delta u$$

(supposing the sum of variable to be between u and $u + \delta u$)

So $$\iiint f(x+y+z)\,x^{l-1}\,y^{m-1}\,z^{n-1}\,dx\,dy\,dz = \frac{\overline{|l}\,\overline{|m}\,\overline{|n}}{\overline{|(l+m+n)}} \int_{h_1}^{h_2} f(u)\,u^{l+m+n-1}\,du$$

Example 35. *Show that* $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3} = \frac{1}{2}\log 2 - \frac{5}{16}$, *the integral being taken throughout the volume bounded by planes* $x = 0,\ y = 0,\ z = 0,\ x + y + z = 1$.

Solution. By Liouville's theorem when $0 < x + y + z < 1$

$$\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3} = \iiint \frac{x^{l-1}y^{l-1}z^{l-1}\,dxdydz}{(x+y+z+1)^3} \qquad (0 \le x+y+z \le 1)$$

$$= \frac{\overline{|1}\,\overline{|1}\,\overline{|1}}{\overline{|1+1+1}} \int_0^1 \frac{1}{(u+1)^3} u^{3-1}\,du = \frac{1}{2}\int_0^1 \frac{u^2}{(u+1)^3}\,du$$

$$= \frac{1}{2}\int_0^1 \left[\frac{1}{u+1} - \frac{2}{(u+1)^2} + \frac{1}{(u+1)^3}\right] du \qquad \text{(Partial fractions)}$$

$$= \frac{1}{2}\left[\log(u+1) + \frac{2}{u+1} - \frac{1}{2(u+1)^2}\right]_0^1$$

$$= \frac{1}{2}\left[\log 2 + 2\left(\frac{1}{2} - 1\right) - \left(\frac{1}{8} - \frac{1}{2}\right)\right] = \frac{1}{2}\log 2 - \frac{5}{16}$$

Proved.

Example 36. *Find the value of* $\iiint \log(x+y+z)\,dx\,dy\,dz$ *the integral extending over all positive and zero values of x, y, z subject to the condition* $0 < x + y + z < 1$.

Solution. By Liouville's theorem when $0 < x + y + z < 1$

$$\iiint \log(x+y+z)\,dx\,dy\,dz$$

$$= \iiint \log(x+y+z)\,x^{1-1}\,y^{1-1}\,z^{1-1}\,dx\,dy\,dz = \frac{\overline{|1}\,\overline{|1}\,\overline{|1}}{\overline{|1+1+1}} \int_0^1 (\log u)\,u^{1+1+1-1}\,du$$

$$= \frac{1}{\overline{|3}} \int_0^1 u^2 \log u\,du = \frac{1}{2}\left[\log u\left(\frac{u^3}{3}\right) - \frac{1}{3}\frac{u^3}{3}\right]_0^1 = \frac{1}{2}\left(-\frac{1}{9}\right) = -\frac{1}{18}$$

Ans.

Example 37. *Evalute* $\iiint \frac{dx_1\,dx_2\ldots dx_n}{\sqrt{1 - x_1^2 - x_2^2 \ldots - x_n^2}}$, *integral being extended to all positive values of the variables for which the expression is real.*

(GBTU 2014)

Solution. $\sqrt{1 - x_1^2 - x_2^2 \ldots x_n^2}$ is real only when $0 < x_1^2 + x_2^2 \ldots x_n^2 < 1$

Hence, the given integral is extended for all positive values of the variables $x_1, x_2 \ldots$ and x_n such that $0 < x_1^2 + x_2^2 + \ldots + x_n^2 < 1$

Let us now put $x_1^2 = u_1$ i.e. $x_1 = u_1^{\frac{1}{2}}$ so that $dx_1 = \frac{1}{2}u_1^{-\frac{1}{2}}\,du_1$

$x_2^2 = u_2$ i.e. $x_2 = u_2^{\frac{1}{2}}$ so that $dx_2 = \frac{1}{2}u_2^{-\frac{1}{2}}\,du_2$

$x_n^2 = u_n$ i.e. $x_n = u_n^{\frac{1}{2}}$ so that $dx_n = \frac{1}{2}u_n^{-\frac{1}{2}}\,du_n$

Making these substitutions, the given condition becomes $0 < u_1 + u_2 + \ldots + u_n < 1$.
Hence, the required integral becomes

$$= \frac{1}{2^n}\iiint \frac{u_1^{-\frac{1}{2}}\cdot u_2^{-\frac{1}{2}}\ldots u_n^{-\frac{1}{2}}\cdot du_1\cdot du_2\ldots du_n}{\sqrt{1-u_1-u_2\ldots-u_n}}$$

$$= \frac{1}{2^n}\iiint \frac{u_1^{\frac{1}{2}-1}\cdot u_2^{\frac{1}{2}-1}\ldots u_n^{\frac{1}{2}-1}\cdot du_1\cdot du_2\ldots du_n}{\sqrt{1-u_1-u_2\ldots-u_n}}$$

$$= \frac{1}{2^n}\frac{\Gamma\frac{1}{2}\cdot\Gamma\frac{1}{2}\ldots\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+\ldots+\frac{1}{2}\right)}\int_0^1 \frac{1}{\sqrt{1-u}}\cdot u^{\left(\frac{1}{2}+\frac{1}{2}+\ldots+\frac{1}{2}\right)-1}\,du$$

By Liouville's Extension of Dirichlet's Theorem

$$= \frac{1}{2^n}\frac{\left(\Gamma\frac{1}{2}\right)^n}{\Gamma\frac{n}{2}}\int_0^1 \frac{1}{\sqrt{1-u}}\cdot u^{\frac{n}{2}-1}\,du$$

$$= \frac{1}{2^n}\frac{\left(\sqrt{\pi}\right)^n}{\Gamma\frac{n}{2}}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-\sin^2\theta}}\cdot(\sin^2\theta)^{\frac{n}{2}-1}\,2\sin\theta\cos\theta\,d\theta \qquad (u = \sin^2\theta)$$

$$= \frac{1}{2^{n-1}}\frac{\left(\sqrt{\pi}\right)^n}{\Gamma\frac{n}{2}}\int_0^{\frac{\pi}{2}} \sin^{n-1}\theta\cos^{o}\theta\,d\theta = \frac{1}{2^{n-1}}\frac{\pi^{\frac{n}{2}}}{\Gamma\frac{n}{2}}\frac{\Gamma\frac{n}{2}\Gamma\frac{1}{2}}{2\Gamma\frac{n+1}{2}} = \frac{1}{2^n}\frac{\pi^{\frac{n+1}{2}}}{\Gamma\frac{n+1}{2}} \qquad \textbf{Ans.}$$

Example 38. *Evaluate* $\iiint \frac{\sqrt{1-x^2-y^2-z^2}}{1+x^2+y^2+z^2}\,dx\,dy\,dz$, *integral being taken over all positive values of x, y, z such that* $0 \le x^2 + y^2 + z^2 \le 1$.

Solution. Putting $x^2 = u$, $y^2 = v$, $z^2 = w$ so that $0 \le u + v + w \le 1$

Also, $x = \sqrt{u} \Rightarrow dx = \frac{1}{2\sqrt{u}}\,du$

$y = \sqrt{v} \Rightarrow dy = \frac{1}{2\sqrt{v}}\,dv$

$$z=\sqrt{w} \qquad \Rightarrow \qquad dz=\frac{1}{2\sqrt{w}}\,dw$$

$\therefore$ The given integral

$$=\iiint\sqrt{\frac{1-(u+v+w)}{1+(u+v+w)}}\,\frac{du\,dv\,dw}{8\sqrt{uvw}}$$

$$=\frac{1}{8}\iiint u^{\frac{1}{2}-1}\,v^{\frac{1}{2}-1}\,w^{\frac{1}{2}-1}\sqrt{\frac{1-(u+v+w)}{1+(u+v+w)}}\,du\,dv\,dw$$

$$=\frac{1}{8}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right)}\int_0^1\sqrt{\frac{1-u}{1+u}}\,u^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-1}\,du \qquad \text{[Using Liouville's extension]}$$

$$=\frac{1}{8}\,\frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^3}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\int_0^1\frac{(1-u)}{\sqrt{1-u^2}}\,u^{1/2}\,du=\frac{\pi}{4}\int_0^1\frac{\left(1-\sqrt{t}\right)}{\sqrt{1-t}}\,t^{1/4}\,\frac{dt}{2\sqrt{t}} \qquad \text{where } u^2=t$$

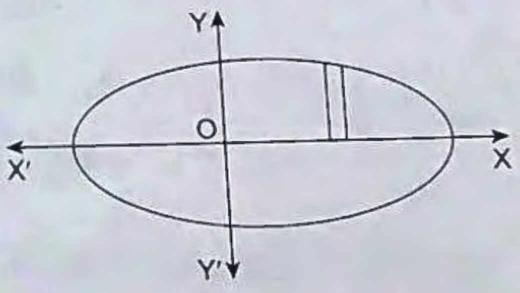
$$=\frac{\pi}{8}\int_0^1\frac{\left(1-\sqrt{t}\right)t^{-1/4}}{\sqrt{1-t}}\,dt=\frac{\pi}{8}\left[\int_0^1 t^{\frac{3}{4}-1}(1-t)^{1/2-1}\,dt-\int_0^1 t^{5/4-1}(1-t)^{1/2-1}\,dt\right]$$

$$=\frac{\pi}{8}\left[\beta\left(\frac{3}{4},\frac{1}{2}\right)-\beta\left(\frac{5}{4},\frac{1}{2}\right)\right]$$

Ans.

Example 39. *Find the area and the mass contained in the first quadrant enclosed by the curve* $\left(\frac{x}{a}\right)^{\alpha}+\left(\frac{y}{b}\right)^{\beta}=1$ *where* $\alpha>0$, $\beta>0$ *given that density at any point* $p(xy)$ *is* $k\sqrt{xy}$ *(U.P. I Semester 2008)*

Solution. Here, we have $\left(\frac{x}{a}\right)^{\alpha}+\left(\frac{y}{b}\right)^{\beta}=1$



Put $\left(\frac{x}{a}\right)^{\alpha}=\cos^2 t$ and $\left(\frac{y}{b}\right)^{\beta}=\sin^2 t$

$\Rightarrow x=a\cos^{\frac{2}{\alpha}}t$ $\qquad\Rightarrow y=b\sin^{\frac{2}{\beta}}t$

$$dx=\frac{2}{\alpha}\,\alpha\cos^{\frac{2}{\alpha}-1}t\,(-\sin t)\,dt$$

$$\text{Area}=\int y\,dx$$

$$=\int_0^{\frac{\pi}{2}}\left(b\sin^{\frac{2}{\beta}}t\right)\left(-\frac{2a}{\alpha}\cos^{\frac{2}{\alpha}-1}t\sin t\right)dt$$

$$= \frac{-2ab}{\alpha}\int_0^{\frac{\pi}{2}} \sin^{\frac{2}{\beta}+1} t \cos^{\frac{2}{\alpha}-1} t\, dt$$

$$= \frac{-2ab}{\alpha}\,\frac{\Gamma\left(\frac{\frac{2}{\beta}+1+1}{2}\right)\Gamma\left(\frac{\frac{2}{\alpha}+1+1}{2}\right)}{2\,\Gamma\left(\frac{\beta+1}{\beta}+\frac{1}{\alpha}\right)}$$

$$= \frac{-ab}{\alpha}\,\frac{\Gamma\left(1+\frac{1}{\beta}\right)\Gamma\left(\frac{1}{\alpha}\right)}{\Gamma\left(\frac{\alpha+\beta+\alpha\beta}{\alpha\beta}\right)}$$

$$= -ab\,\frac{\frac{1}{\alpha}\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(1+\frac{1}{\beta}\right)}{\Gamma\left(\frac{\alpha+\beta+\alpha\beta}{\alpha\beta}\right)}$$

$$= -ab\,\frac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma\left(1+\frac{1}{\beta}\right)}{\Gamma\left(\frac{\alpha+\beta+\alpha\beta}{\alpha\beta}\right)}$$

Required area $= \dfrac{\Gamma\left(\frac{\alpha+1}{\alpha}\right)\Gamma\left(\frac{\beta+1}{\beta}\right)}{\Gamma\left(\frac{\alpha+\beta+\alpha\beta}{\alpha\beta}\right)}$ Ans.

Density $= k\sqrt{xy}$

Mass = Volume × Density

density $= \dfrac{\text{Mass}}{\text{Volume}}$

$$= \int\left[y\,dx\,k\sqrt{xy}\right]$$

$$= 4k\int_0^{\frac{\pi}{2}} x^{\frac{1}{2}} y^{\frac{3}{2}}\,dx$$

$$= 4k\int_0^{\frac{\pi}{2}} (a\cos^{\frac{2}{\alpha}} t)^{\frac{1}{2}} (b\sin^{\frac{2}{\beta}} t)^{\frac{3}{2}}\,\frac{2}{\alpha}a\cos^{\frac{2}{\alpha}-1} t(-\sin t)\,dt$$

$$= 4k\int_0^{\frac{\pi}{4}} a^{\frac{1}{2}}\cos^{\frac{1}{\alpha}} t\, b^{\frac{3}{2}}\sin^{\frac{3}{\beta}} t\,\frac{2}{\alpha}a\cos^{\frac{2}{\alpha}-1} t(-\sin t\,dt)$$

$$= \frac{8k}{\alpha}a^{\frac{3}{2}}b^{\frac{3}{2}}\int_0^{\frac{\pi}{2}}\sin^{\frac{3}{\beta}+1} t\cos^{\frac{2}{\alpha}-1} t\,dt$$ (−ve sign to be neglected)

$$= \frac{8k}{\alpha} a^{\frac{3}{2}} b^{\frac{3}{2}} \frac{\Gamma\left(\frac{\frac{3}{\beta}+1+1}{2}\right)\Gamma\left(\frac{\frac{2}{\alpha}+1+1}{2}\right)}{\Gamma\left(\frac{7}{4}\right)} = \frac{8k}{\alpha} a^{\frac{1}{2}} b^{\frac{3}{2}} \beta\left(\frac{5}{2}, \frac{2}{\alpha}-1\right)$$

$$= \frac{8k}{\alpha} a^{\frac{3}{2}} b^{\frac{3}{2}} \frac{\Gamma\left(\frac{\frac{\beta+1}{\beta}}{2}\right)\Gamma\left(\frac{1}{\alpha}\right)}{\Gamma\left(\frac{\beta+1}{\beta}+\frac{1}{\alpha}\right)}$$ **Ans.**

Example 40. *Find the mass of an octant of the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, *the density at any point being* $\rho = k\,x\,y\,z$

(U.P., I Semester, 2009, 2015)

Solution. $\text{Mass} = \iiint \rho\, dv = \iiint (k\,x\,y\,z)\, dx\, dy\, dz$

$$= k\iiint (x\,dx)(y\,dy)(z\,dz) \qquad \ldots (1)$$

Putting $\frac{x^2}{a^2}=u,\ \frac{y^2}{b^2}=v,\ \frac{z^2}{c^2}=w$ and $u+v+w=1$

So that $\frac{2x\,dx}{a^2}=du, \frac{2y\,dy}{b^2}=dv, \frac{2z\,dz}{c^2}=dw$

$$\text{Mass} = k\iiint \left(\frac{a^2\,du}{2}\right)\left(\frac{b^2\,dv}{2}\right)\left(\frac{c^2\,dw}{2}\right)$$

$$= \frac{ka^2b^2c^2}{8}\iiint du\,dv\,dw, \qquad \text{where } 0 < u+v+w \le 1.$$

$$= \frac{ka^2b^2c^2}{8}\iiint u^{1-1}\, v^{1-1}\, w^{1-1}\, du\,dv\,dw$$

$$= \frac{ka^2b^2c^2}{8}\frac{\Gamma 1\,\Gamma 1\,\Gamma 1}{\Gamma(3+1)} = \frac{ka^2b^2c^2}{8\times 6} = \frac{ka^2b^2c^2}{48}$$ **Ans.**

Example 41. *Find the mass of a solid* $\left(\frac{x}{a}\right)^p+\left(\frac{y}{b}\right)^q+\left(\frac{z}{c}\right)^r=1$, *the density at any point being* $\rho = kx^{l-1}\,y^{m-1}\,z^{n-1}$, *where x, y, z are all positive.* (U.P., I Jan. 2011)

Solution. Here, we have

$$\left(\frac{x}{a}\right)^p+\left(\frac{y}{b}\right)^q+\left(\frac{z}{c}\right)^r=1$$

$$\text{Density} = kx^{l-1}\,y^{m-1}\,z^{n-1}$$

Mass = Volume × Density

$$= \iiint dx\,dy\,dz \cdot kx^{l-1}y^{m-1}z^{n-1} \qquad \ldots (1)$$

Put $\left(\frac{x}{a}\right)^p = u \Rightarrow x = au^{\frac{1}{p}} \Rightarrow dx = \frac{a}{p}u^{\frac{1}{p}-1}\,du$

$$\left(\frac{y}{b}\right)^q = v \Rightarrow y = bv^{\frac{1}{q}} \Rightarrow dy = \frac{b}{q}v^{\frac{1}{q}-1}\,dv$$

$$\left(\frac{z}{c}\right)^r = w \Rightarrow z = cw^{\frac{1}{r}} \Rightarrow dz = \frac{c}{r}w^{\frac{1}{r}-1}\,dw$$

Then, $u + v + w \le 1$. Also $u \ge 0,\ v \ge 0,\ w \ge 0.$

Since x, y, z are all positive

Putting above values in (1), we get

$$\text{Mass} = k\iiint (au^{\frac{1}{p}})^{l-1}(bv^{\frac{1}{q}})^{m-1}(cw^{\frac{1}{r}})^{n-1}\left[\frac{a}{p}u^{\frac{1}{p}-1}\,du\right]\left[\frac{b}{q}v^{\frac{1}{q}-1}\,dv\right]\left[\frac{c}{r}w^{\frac{1}{r}-1}\,dw\right]$$

$$= k\frac{a^l b^m c^n}{pqr}\iiint u^{\frac{l}{p}-1}\,v^{\frac{m}{q}-1}\,w^{\frac{n}{r}-1}\,du\,dv\,dw = k\frac{a^l b^m c^n}{pqr}\,\frac{\Gamma\left(\frac{l}{p}\right)\Gamma\left(\frac{m}{q}\right)\Gamma\left(\frac{n}{r}\right)}{\Gamma\left(\frac{l}{p}+\frac{m}{q}+\frac{n}{r}+1\right)} \qquad \textbf{Ans.}$$

Example 42. *Evaluate* $I = \iiint_V x^{\alpha-1}y^{\beta-1}z^{\gamma-1}\,dx\,dy\,dz$, *where V is the region in the first octant bounded by sphere* $x^2 + y^2 + z^2 = 1$ *and the coordinate planes.*

Solution. Let

$$x^2 = u \Rightarrow x = \sqrt{u} \qquad \therefore\ dx = \frac{1}{2\sqrt{u}}\,du$$

$$y^2 = v \Rightarrow y = \sqrt{v} \qquad \therefore\ dy = \frac{1}{2\sqrt{v}}\,dv$$

$$z^2 = w \Rightarrow z = \sqrt{w} \qquad \therefore\ dz = \frac{1}{2\sqrt{w}}\,dw$$

Then, $u + v + w = 1$. Also, $u \ge 0,\ v \ge 0,\ w \ge 0.$

$$I = \iiint_V (\sqrt{u})^{\alpha-1}(\sqrt{v})^{\beta-1}(\sqrt{w})^{\gamma-1}\,\frac{du}{2\sqrt{u}}\cdot\frac{dv}{2\sqrt{v}}\cdot\frac{dw}{2\sqrt{w}}$$

$$= \frac{1}{8}\iiint u^{(\alpha/2)-1}\,v^{(\beta/2)-1}\,w^{(\gamma/2)-1}\,du\,dv\,dw$$

$$= \frac{1}{8}\,\frac{\Gamma(\alpha/2)\,\Gamma(\beta/2)\,\Gamma(\gamma/2)}{\Gamma\left((\alpha/2)+(\beta/2)+(\gamma/2)+1\right)}$$

EXERCISE 7.3

Evaluate:

1. $\iiint e^{x+y+z}\,dx\,dy\,dz$ taken over the positive octant such that $x+y+z \le 1$. **Ans.** $\frac{e-2}{2}$

2. $\iiint \frac{dx\,dy\,dz}{(a^2-x^2-y^2-z^2)}$ for all positive values of the variables for which the expression is real.

   a^2

   [**Hint.** $a^2-x^2-y^2-z^2>0 \Rightarrow 0<x^2+y^2+z^2<a^2$] **Ans.** $\frac{\pi^2 a^2}{8}$

3. $\iiint_R (x+y+z+1)^2\,dx\,dy\,dz$ where R is defined by $x \ge 0, y \ge 0, z \ge 0, x+y+z \le 1$. **Ans.** $\frac{31}{60}$

4. $\iiint x^{-\frac{1}{2}} y^{-\frac{1}{2}} z^{-\frac{1}{2}} (1-x-y-z)^{\frac{1}{2}}\,dx\,dy\,dz,\ x+y+z \le 1, x>0, y>0, z>0$ **Ans.** $\frac{\pi^2}{4}$

5. Show that $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^2} = \frac{3}{4} - \log 2$, the integral being taken throughout the volume bounded by the planes $x=0, y=0, z=0$ and $x+y+z=1$.

Very Short Answer Type Questions

2 Marks:

1. What is the value of $\left|\overline{\frac{1}{2}}\right.$ (*UPTU, 2010*) **Ans.** $\sqrt{\pi}$

2. Evaluate $\int_0^\infty e^{-x} x^{n-1}\,dx$. **Ans.** $\overline{|n}$

3. Evaluate $\int_0^\infty \sqrt{x} e^{-x}\,dx$. (*GBTU, 2014*) **Ans.** $\frac{\sqrt{x}}{2}$

4. Evaluate $\int_0^\infty e^{-x} x^4\,dx$ **Ans.** $\sqrt{5}$

5. Find the value of $\left|\overline{-\frac{5}{2}}\right.$ **Ans.** $\frac{-8}{15}\sqrt{\pi}$

6. Find the value of $\int_0^1 x^{l-1}(1-x)^{m-1}\,dx$ **Ans.** $\beta\,(l, m)$

7. Write the value of $\beta\,(l, m)$ **Ans.** $\frac{\overline{|l}\,\overline{|m}}{\overline{|l+m}}$

8. Find the value of $\left|\overline{\frac{1}{4}}\right. \cdot \left|\overline{\frac{4}{4}}\right.$ **Ans.** $\pi\sqrt{2}$

9. Write the formula $\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta\,d\theta$ of **Ans.** $\frac{\overline{\left|\left(\frac{p+1}{2}\right)\right.}\,\overline{\left|\left(\frac{q+1}{2}\right)\right.}}{2\,\overline{\left|\left(\frac{p+q+1}{2}\right)\right.}}$

10. Evaluate $\int_0^{\frac{\pi}{2}} \sin^2\theta \cos^4\theta d\theta$.

Ans. $\frac{\pi}{32}$

11. Evaluate $\int_0^{\frac{\pi}{2}} \sin^{10} x\, dx$.

Ans. $\frac{\Gamma\left(\frac{11}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(6)}$

12. Write the Duplicate formula:

Ans. $\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$, where m is positive.

13. Write the Dirichlet integral.

Ans. $\iiint\limits_V x^{l-1}y^{m-1}z^{n-1}dx\, dy\, dz = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}$

where V is the region $x \geq 0$, $y \geq 0$, $z \geq 0$ and $0 < x + y + z \leq 1$.

14. Evaluate $\iiint x^{l-1}y^{m-1}x^{n-1}dx\, dy\, dz$.

Ans. $\frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}$

15. Find the value of $\beta\left(\frac{1}{2}, 2\right)$.

Ans. $\frac{4}{3}$